Quantum Field Theory 2 – Problem set 1

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Suggested reading before solving these problems: Chapter 1 in the script and/or chapter 9.1-9.3 in *Peskin & Schroeder*.

Problem 1: Asymptotic series

Consider the following function

$$Z(\lambda) = \int_{-\infty}^{\infty} d\varphi \; e^{-\frac{1}{2}\varphi^2 - \frac{\lambda}{4}\varphi^4}. \tag{1}$$

a) Calculate the coefficients Z_n of the perturbative series in λ

$$Z = \sum_{n=0}^{\infty} Z_n \lambda^n. \tag{2}$$

(You may use $\int_0^\infty dt\; e^{-t} t^x = \Gamma(x+1)$.)

b) The error of the partial sum with order N can can be estimated by an upper bound according to

$$\begin{aligned} R_N &= |Z(\lambda) - \sum_{n=0}^{N} Z_n \lambda^n| \\ &= \int d\varphi\; e^{-\frac{1}{2}\varphi^2} \left| e^{-\frac{1}{4}\lambda\varphi^4} - \sum_{n=0}^{N} \frac{1}{n!}\left(-\frac{1}{4}\lambda\varphi^4\right)^n \right| \\ &\leq \int d\varphi\; e^{-\frac{1}{2}\varphi^2} \frac{1}{(N+1)!}\left(\frac{1}{4}\lambda\varphi^4\right)^{N+1} = \lambda^{N+1}|Z_{N+1}|. \end{aligned}$$

Use Stirling's formula $\Gamma(x) \to x^{x-\frac{1}{2}} e^{-x}\sqrt{2\pi}$ $(x \to \infty)$ to estimate $\lambda^n Z_n$ for large n and determine the (approximate) order $N = N_{\min}$ where R_N is minimal.

Problem 2: Generating functions

Consider a probabilistic theory for an N-dimensional vector $\mathbf{x}$. Expectation values are defined as

$$\langle \dots \rangle = \frac{\int d\mathbf{x}\; (\dots)\; e^{-S(\mathbf{x})}}{\int d\mathbf{x}\; e^{-S(\mathbf{x})}}.$$

with the "action"

$$S(\mathbf{x}) = \frac{1}{2}P_{ab}\, x_a x_b + \frac{1}{3!}\gamma_{abc}\, x_a x_b x_c + \frac{1}{4!}\lambda_{abcd}\, x_a x_b x_c x_d + \dots$$

It is useful to define the partition function

$$Z(\mathbf{J}) = \int d\mathbf{x}\; e^{-S(\mathbf{x}) + J_a x_a}.$$

a) Show that

$$\langle x_{i_1} \dots x_{i_L} \rangle = \frac{1}{Z(\mathbf{J})} \frac{\partial}{\partial J_{i_1}} \cdots \frac{\partial}{\partial J_{i_L}} Z(\mathbf{J}) \bigg|_{\mathbf{J}=0}. \tag{3}$$

b) Show that one can write formally

$$Z(\mathbf{J}) = \sqrt{\frac{(2\pi)^N}{\det P}}\; e^{-\left(\frac{1}{3!}\gamma_{abc} \frac{\partial}{\partial J_a}\frac{\partial}{\partial J_b}\frac{\partial}{\partial J_c} + \frac{1}{4!}\lambda_{abcd}\frac{\partial}{\partial J_a}\frac{\partial}{\partial J_b}\frac{\partial}{\partial J_c}\frac{\partial}{\partial J_d} + \dots\right)}\; e^{\frac{1}{2}J_a P^{-1}_{ab} J_b}.$$

c) Derive expressions for the correlation functions $\langle x_a \rangle$, $\langle x_a x_b \rangle$, $\langle x_a x_b x_c \rangle$ and $\langle x_a x_b x_c x_d \rangle$ to linear order in γ and λ. (Higher terms in $S(\mathbf{x})$ are neglected.) Can you draw corresponding Feynman diagrams?

d) The connected correlation functions are given by

$$\langle x_{i_1} \dots x_{i_L} \rangle_c = \frac{\partial}{\partial J_{i_1}} \cdots \frac{\partial}{\partial J_{i_L}} W(\mathbf{J}) \bigg|_{\mathbf{J}=0}, \tag{4}$$

with

$$W(\mathbf{J}) = \ln Z(\mathbf{J}). \tag{5}$$

Derive expressions for $\langle x_a \rangle_c$, $\langle x_a x_b \rangle_c$, $\langle x_a x_b x_c \rangle_c$ and $\langle x_a x_b x_c x_d \rangle_c$ to linear order in γ and λ. The generating function $W(\mathbf{J})$ of the connected correlation functions is also known as the Schwinger function.